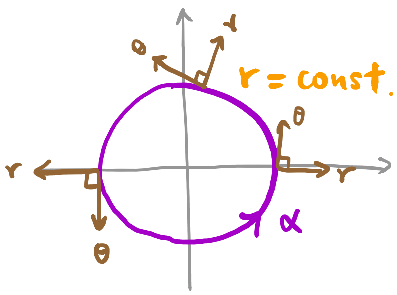
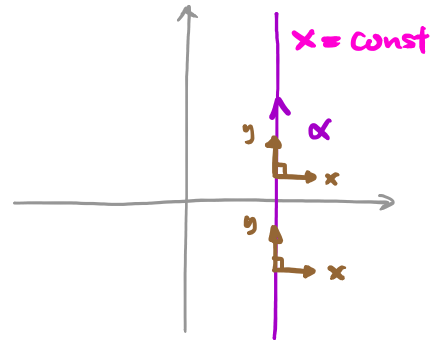
§ Local theory of plane curves

Consider two plane curves:

(r,θ): polar coordinates

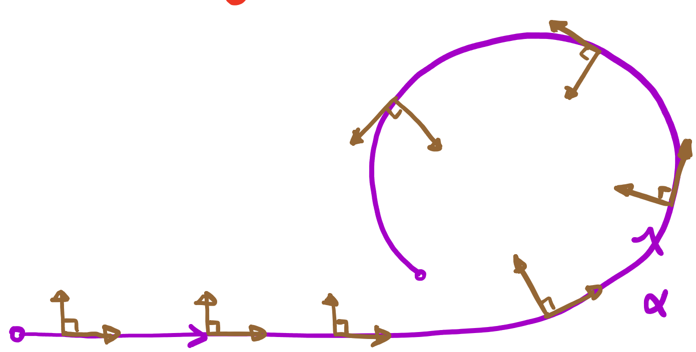


(x,y): rectangular coordinates



Question: What is the "best" coordinate system on a given (regular) curve?

E.g.

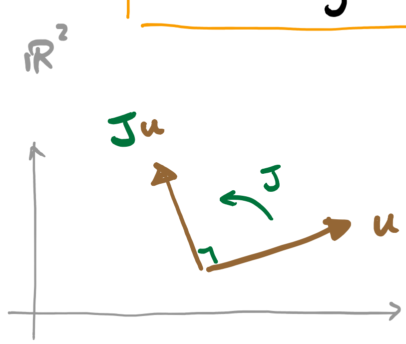


depends on where we are on the curve

$\Downarrow$

"moving frame"

Let $\boxed{J = \text{counterclockwise rotation by } 90^\circ \text{ in } \mathbb{R}^2 \ (\cong \mathbb{C})}$ special in dim. 2



Given any unit vector $u \in \mathbb{R}^2$

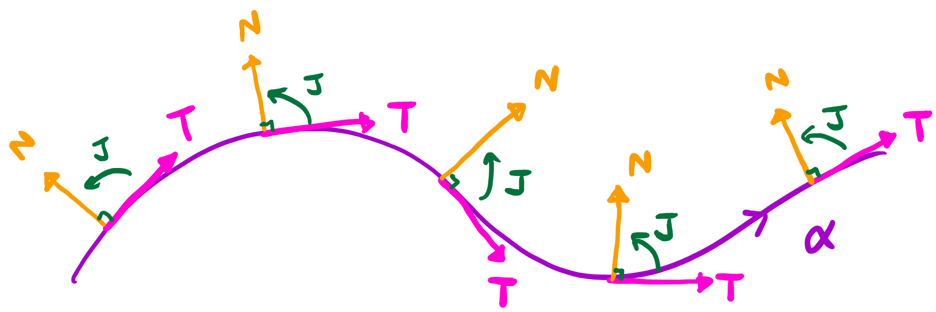
$\{u, Ju\}$ is a (pos. oriented) orthonormal basis (O.N.B.)

called a **frame**.

Definition: Let $\alpha : I \to \mathbb{R}^2$ be a plane curve p.b.a.l.

Define: $\begin{cases} T(s) = \alpha'(s) & \text{unit tangent} \\ N(s) = J(T(s)) & \text{unit normal} \end{cases}$

$\{T(s), N(s)\}$ Frenet frame along α



Note: $\{T(s), N(s)\}$ O.N.B for each $s \in I$

$$\Rightarrow \begin{cases} \langle T(s), T(s) \rangle \equiv 1 \equiv \langle N(s), N(s) \rangle & \cdots\cdots \text{①} \\ \langle T(s), N(s) \rangle \equiv 0 & \cdots\cdots\cdots\cdots\cdots \text{②} \end{cases}$$

Differentiate ① :

$$\langle T'(s), T(s) \rangle \equiv 0 \equiv \langle N'(s), N(s) \rangle$$

Differentiate ② : by product rule

$$\langle T'(s), N(s) \rangle + \langle T(s), N'(s) \rangle \equiv 0$$

Hence,

$$\begin{cases} T'(s) = k(s) N(s) \\ N'(s) = -k(s) T(s) \end{cases}$$

Definition: Let $\alpha : I \to \mathbb{R}$ be a curve p.b.a.l. with Frenet frame $\{T(s), N(s)\}$.

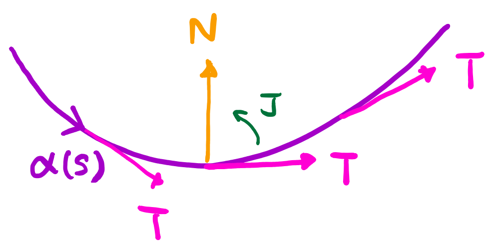
$$\boxed{k(s) := \langle T'(s), N(s) \rangle}$$

curvature of α (at s)

Remark: 1) $k : I \to \mathbb{R}$ is a smooth function

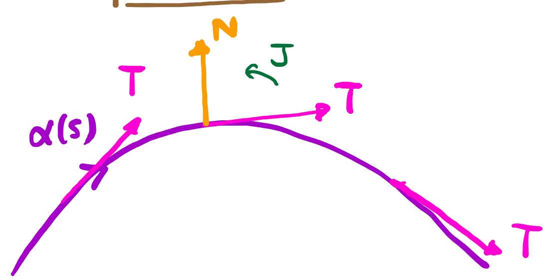
2) k has a sign:

$\boxed{k > 0}$



"T turning towards N"

$\boxed{k < 0}$



"T turning away from N"

Caution: No such interpretation for space curves!

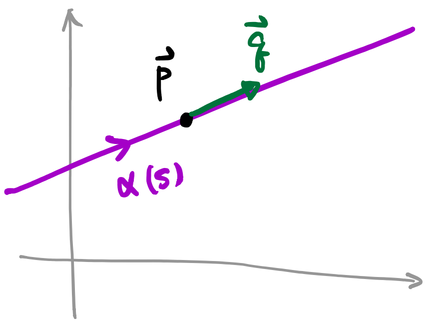
3) If $\alpha : I \to \mathbb{R}^2$ is NOT p.b.a.l. (but regular)

then "reparametrize" by $\beta = \alpha \circ \phi : J \to \mathbb{R}^2$ p.b.a.l.

define $\boxed{k_\alpha(t) := k_\beta(\phi^{-1}(t))}$

$\Rightarrow$ Note: Curvature is invariant under reparametrization!

Example 1 : Straight line



$\boxed{k \equiv 0}$

$\alpha(s) = \vec{p} + s \cdot \vec{q}$, $s \in \mathbb{R}$

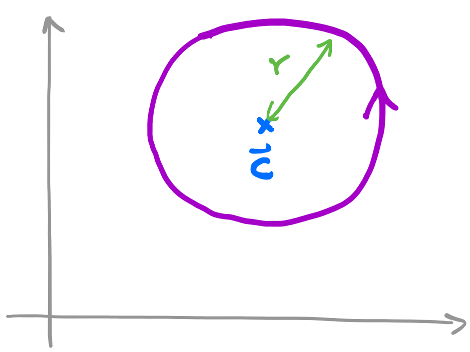
α is p.b.a.l. $\Leftrightarrow$ $|\vec{q}| = 1$.

Frenet frame :

$$\begin{cases} T(s) = \alpha'(s) = \vec{q} & \text{constant!} \\ N(s) = J\,T(s) = J\vec{q} \end{cases}$$

$k(s) = \langle T'(s), N(s) \rangle \equiv 0$.

Example 2 : Circles



$\boxed{k = \frac{1}{r}}$

" Bigger circles have smaller curvature "

$\alpha(s) = \vec{c} + r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$

p.b.a.l.

Frenet frame :

$$\begin{cases} T(s) = \alpha'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right) \\ N(s) = J\,T(s) = \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right) \end{cases}$$

Curvature :

$$\begin{aligned} k(s) &= \langle T'(s), N(s) \rangle \\ &= \left\langle \frac{1}{r}\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right), \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right) \right\rangle \\ &= \frac{1}{r} \qquad \text{constant!} \end{aligned}$$

Exercise: Repeat the above calculation using a clockwise parametrization.

Proposition: Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l.
If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a rigid motion, then

$$\beta = \varphi \circ \alpha : I \to \mathbb{R}^2 \quad \text{is also p.b.a.l.}$$

and

$$k_\beta(s) = \begin{cases} k_\alpha(s) & \text{if } \varphi \text{ is orientation-preserving} \\ -k_\alpha(s) & \text{if } \varphi \text{ is orientation-reversing} \end{cases}$$

Proof: Exercise!

"Curvature is a geometric quantity"

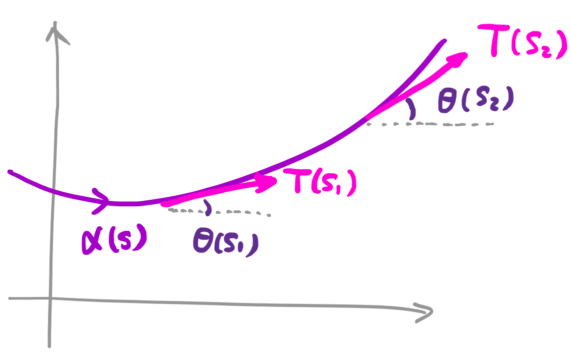
(not necessarily p.b.a.l.)

Exercise: If $\alpha : I \to \mathbb{R}^2$ is a regular plane curve, show that

$$\boxed{k_\alpha(t) = \frac{\det(\alpha'(t), \alpha''(t))}{|\alpha'(t)|^3}}$$

Exercise: Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l.

Denote $\theta(s)$ = angle of $T(s)$ measured from x-axis.



Then: $\boxed{\theta'(s) = k(s)}$

"Curvature measures the rate of turning of the unit tangent vector"

Exercise: The only plane curves with $k \equiv$ constant are straight lines and circles.

$(k \equiv 0)$ $\qquad$ $(k \equiv \pm \frac{1}{r})$

depends on orientation

Question: In general, does the curvature $k : I \to \mathbb{R}$ determine the curve $\alpha : I \to \mathbb{R}^2$ (p.b.a.l.) "completely" (up to rigid motions)?

YES!

Fundamental Theorem of Plane Curves

Given a smooth function $k : I \to \mathbb{R}$, $\exists$ $\alpha : I \longrightarrow \mathbb{R}^2$ p.b.a.l. (defined on the same I)

s.t. $\qquad k_\alpha(s) = k(s) \quad \forall s \in I$

Moreover, α is unique up to orientation-preserving rigid motions.

Note: The basic idea is that $k_\alpha \approx \alpha''$ (but non-linear!)

$$k_\alpha \approx \alpha'' \xrightarrow[\text{integrate}]{} \alpha' = T \xrightarrow[\text{integrate}]{} \alpha$$

ambiguity by $\varphi = A\vec{x} + b$ "integration constants"

Proof: (I) Existence

Fix $s_0 \in I$, define (Recall: $\theta' = k$)

$$\theta(s) := \int_{s_0}^{s} k(u)\,du$$

hence if we set

$$\alpha'(s) = T(s) = \underbrace{(\cos\theta(s), \sin\theta(s))}^{\text{unit vector}}$$

Integrating gives

$$\alpha(s) = \left(\int_{s_0}^{s} \cos\theta(t)\,dt \, , \int_{s_0}^{s} \sin\theta(t)\,dt \right)$$

<u>Exercise</u>: Check α is p.b.a.l. and $k_\alpha(s) = k(s)$.

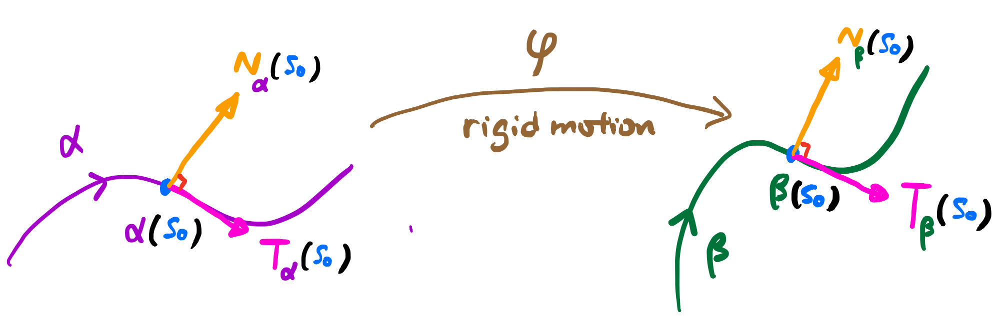
(II) Uniqueness

Suppose $\beta : I \to \mathbb{R}^2$ is another curve p.b.a.l. s.t.

$$k_\beta(s) = k(s) = k_\alpha(s). \quad \forall s \in I.$$

Fix $s_0 \in I$. Consider the Frenet frames

$$\{T_\alpha(s), N_\alpha(s)\} \quad \text{and} \quad \{T_\beta(s), N_\beta(s)\}$$



$\exists$ unique orientation-preserving rigid motion

$$\varphi(x) = Ax + b$$

s.t. (1) $\varphi(\alpha(s_0)) = \beta(s_0)$ "match the point"

(2) $\begin{cases} A(T_\alpha(s_0)) = T_\alpha(s_0) \\ A(N_\beta(s_0)) = N_\beta(s_0) \end{cases}$ "match the frame"

Claim: $\varphi \circ \alpha = \beta$

Consider $f(s) = |T_{\varphi\circ\alpha}(s) - T_\beta(s)|^2$, $s \in I$.

$$= |A\,T_\alpha(s) - T_\beta(s)|^2$$

Differentiate in s, applying Frenet equations,

$$\frac{1}{2} f' = \langle A T_\alpha' - T_\beta', A T_\alpha - T_\beta \rangle$$

$$= \langle A(k_\alpha N_\alpha) - k_\beta N_\beta, A T_\alpha - T_\beta \rangle$$

$(k_\alpha = k = k_\beta)$ $$= k \langle A N_\alpha - N_\beta, A T_\alpha - T_\beta \rangle$$

$$= k\big(\langle A N_\alpha, A T_\alpha \rangle + \langle N_\beta, T_\beta \rangle - \langle A N_\alpha, T_\beta \rangle - \langle N_\beta, A T_\alpha \rangle \big)$$

Note: • $\langle N_\beta, T_\beta \rangle = 0$ since $N_\beta \perp T_\beta$

• $\langle AN_\alpha, AT_\alpha \rangle \underset{\substack{\uparrow \\ \because A \in SO(2)}}{=} \langle N_\alpha, T_\alpha \rangle = 0$

• $$\begin{aligned} \langle AN_\alpha, T_\beta \rangle &= \langle AJT_\alpha, T_\beta \rangle \\ &= \langle JAT_\alpha, T_\beta \rangle && (\because AJ = JA) \text{ why?} \\ &= \langle J^2AT_\alpha, JT_\beta \rangle && (\because J \in SO(2)) \\ &= -\langle AT_\alpha, N_\beta \rangle && (\because J^2 = -1) \end{aligned}$$

Combining these calculations, we have

$$f'(s) \equiv 0 \quad \forall s \in I$$

Since $f(s_0) = 0$ by the choice of φ, $f(s) \equiv 0 \ \forall s \in I$.

Therefore,

$$(\varphi \circ \alpha)'(s) = T_{\varphi \circ \alpha}(s) = T_\beta(s) = \beta'(s) \qquad \forall s \in I$$

Integrating s and using $\varphi(\alpha(s_0)) = \beta(s_0)$,

$$\Rightarrow \qquad (\varphi \circ \alpha)(s) = \beta(s) \qquad \forall s \in I.$$

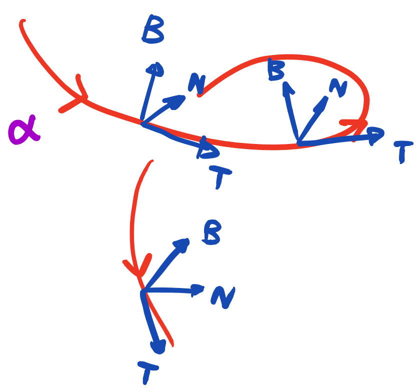
This completes the proof!

$\square$

§ Space curves

Consider now a space curve

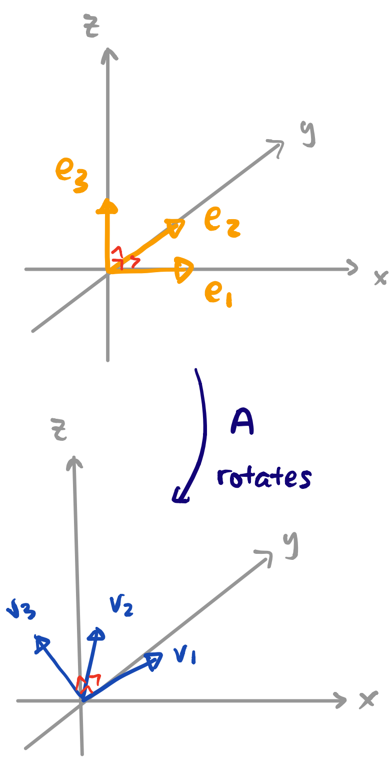
$$\alpha : I \longrightarrow \mathbb{R}^3 \qquad \text{p.b.a.l.}$$



<u>Goal</u>: Define a "moving frame" along α whose rate of change reflects the (extrinsic) "geometry" of α

Frenet frame : $\{T, N, B\}$

Recall first about "frames in $\mathbb{R}^3$"



$\{e_1, e_2, e_3\}$ = standard "frame"

$(e_1 \times e_2 = e_3)$ positively oriented O.N.B.

<u>Linear algebra fact</u>

Given any "frame" $\{v_1, v_2, v_3\}$,

$\exists$ unique $A \in SO(3)$ s.t. $A e_i = v_i$, $i = 1,2,3$

$$SO(3) = \{ A \in M_{3\times 3}(\mathbb{R}) : A^T A = I,\ \det A = 1 \}$$

"Space of frames in $\mathbb{R}^3$"

Recall for plane curves : $\alpha : I \to \mathbb{R}^2$ p.b.a.l.

Frenet frame

$$\{T, N\}$$

$$T = \alpha', \quad N = JT$$

Frenet equations

$$\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

$k := \langle T', N \rangle$ <u>curvature</u>

<u>Note</u>: $k = \pm |T'|$

Now, for a space curve : $\alpha : I \to \mathbb{R}^3$ p.b.a.l.

Define : $T(s) := \alpha'(s)$ tangent

and $\boxed{k(s) := |T'(s)|}$ <u>curvature</u>

<u>Note</u>: $k \geq 0$ for space curves

$\boxed{\text{Assume : } k(s) \neq 0}$ $(*)$

Then, we can define :

$$N(s) := \frac{T'(s)}{|T'(s)|} \qquad \text{normal}$$

and $B(s) := T(s) \times N(s)$ binormal

For any $\alpha : I \to \mathbb{R}^3$ p.b.a.l. satisfying $(*)$ for all $s \in I$, we have defined smoothly along α the

Frenet frame : $\{T(s), N(s), B(s)\}$

Frenet equations: $\alpha : I \to \mathbb{R}^3$ p.b.a.l., $k(s) > 0 \ \forall s \in I$

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \quad \ldots\ldots \ (\#)$$

where $\boxed{\begin{array}{l} k := |T'| \\ \tau := \langle B', N \rangle \end{array}}$ $\begin{array}{l} \text{curvature} \\ \text{torsion} \end{array}$

Note: $k \geq 0$ always, but τ can be < 0, $= 0$ or > 0.

Proof of (#): Use $\{T(s), N(s), B(s)\}$ is O.N.B. $\forall s \in I$

Differentiating w.r.t. s:

$$\left.\begin{array}{l} \langle T, T \rangle \equiv 1 \ \Rightarrow \ \langle T', T \rangle = 0 \\ \langle N, N \rangle \equiv 1 \ \Rightarrow \ \langle N', N \rangle = 0 \\ \langle B, B \rangle \equiv 1 \ \Rightarrow \ \langle B', B \rangle = 0 \end{array}\right\} \Rightarrow \text{diagonal entries in (\#)} = 0.$$

By def^{n} $N = \dfrac{T'}{|T'|} = \dfrac{T'}{\underset{\neq 0}{k}} \Rightarrow \boxed{T' = kN}$

By def^{n} $B = T \times N \Rightarrow B' = T' \times N + T \times N'$

$$= \underbrace{kN \times N}_{= 0} + \underbrace{T \times N'}_{\perp T}$$

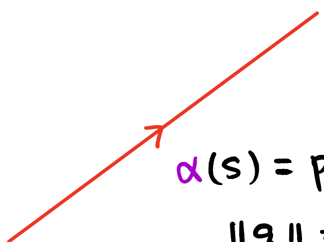
$$\Rightarrow \boxed{B' = \tau N}$$

Finally,

$$\left.\begin{aligned} \langle N, T\rangle \equiv 0 &\Rightarrow \langle N', T\rangle + \underbrace{\langle N, T'\rangle}_{= k} = 0 \\ \langle N, B\rangle \equiv 0 &\Rightarrow \langle N', B\rangle + \underbrace{\langle N, B'\rangle}_{= \tau} = 0 \end{aligned}\right\} \Rightarrow \boxed{N' = -kT - \tau B}$$

$\square$

(Bad) Example 0: Straight lines



$\alpha(s) = p + sq, \quad s \in \mathbb{R}$

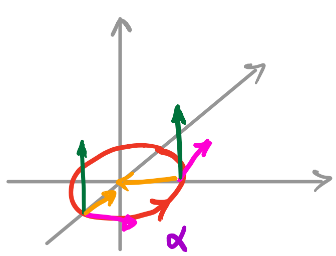
$\|q\| = 1 \quad$ p.b.a.l.

$T(s) = \alpha'(s) = q$

$T'(s) = 0$

$$\boxed{\begin{aligned} &k \equiv 0 \\ &\tau \text{ not defined} \end{aligned}}$$

Example 1: Circles



p.b.a.l.

$\alpha(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right), \quad s \in \mathbb{R}$

$T(s) = \alpha'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right)$

$T'(s) = \frac{1}{r}\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right)$

$\therefore \; k(s) = |T'(s)| = \frac{1}{r} > 0$

$N(s) = \frac{T'(s)}{|T'(s)|} = \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right)$

$B(s) = T(s) \times N(s) = (0, 0, 1)$

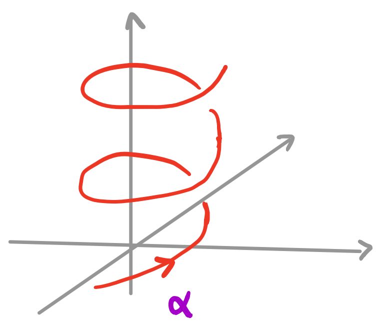
$\therefore \; \tau(s) = \langle B'(s), N(s)\rangle = 0$

$$\boxed{\begin{aligned} &k \equiv \frac{1}{r} \\ &\tau \equiv 0 \end{aligned}}$$

Exercise: Let $\alpha : I \to \mathbb{R}^3$, p.b.a.l., $k > 0$. Then

"α lies on a plane $P \subseteq \mathbb{R}^3$" $\iff$ $\tau \equiv 0$.

Example 2: Helix



p.b.a.l.

$$\alpha(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right), \ s \in \mathbb{R}$$

$$T(s) = \alpha'(s) = \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right)$$

$$T'(s) = \frac{1}{2}\left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$\therefore \ k(s) = |T'(s)| = \frac{1}{2}.$$

$$N(s) = \frac{T'(s)}{|T'(s)|} = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$B(s) = T(s) \times N(s) = \frac{1}{\sqrt{2}}\left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 1\right)$$

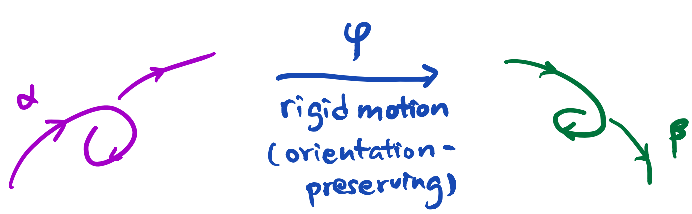
$$B'(s) = \frac{1}{2}\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right)$$

$$\therefore \ \tau(s) = \langle B'(s), N(s) \rangle = -\frac{1}{2}$$

$$\boxed{\begin{aligned} k &\equiv \tfrac{1}{2} \\ \tau &\equiv -\tfrac{1}{2} \end{aligned}}$$

Exercise: k, τ are "geometric" quantity, i.e. they are invariant under orientation-preserving rigid motions of $\mathbb{R}^3$.

$$k_\alpha = k_\beta, \quad \tau_\alpha = \tau_\beta$$



Fundamental Theorem of Space Curves

Given smooth functions $k, \tau : I \to \mathbb{R}$ with $k > 0$,

there exists a space curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l.

s.t. $k_\alpha = k$ and $\tau_\alpha = \tau$

Moreover, α is unique up to orientation-preserving rigid motions of $\mathbb{R}^3$.

Proof: Fix $s_0 \in I$, and any frame $\{T_0, N_0, B_0\}$ of $\mathbb{R}^3$.

Consider the Frenet equations

$$(\#): \begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}}_{\text{given}} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

1st order system linear ODEs

By fundamental existence theorem of ODEs,

$\exists$ solution $T(s), N(s), B(s)$, $s \in I$ to $(\#)$

with "initial condition": $\{T(s_0), N(s_0), B(s_0)\} = \{T_0, N_0, B_0\}$

Claim: $\{T(s), N(s), B(s)\}$ is a frame $\forall s \in I$.

Proof: Define a 3×3 matrix

$$M(s) = \begin{pmatrix} - & T(s) & - \\ - & N(s) & - \\ - & B(s) & - \end{pmatrix}$$

It suffices to check:

$M(s) \in SO(3) \quad \forall s \in I$.

Known: $M(s_0) \in SO(3)$ since $\{T_0, N_0, B_0\}$ is a frame.

Define $Q = M M^T$ (depends on $s \in I$)

Check: Q satisfies the following ODE :

$$(*) \begin{cases} Q' = KQ - QK \\ Q(s_0) = I \end{cases}$$

We can rewrite the Frenet equations (#) in

matrix form : $\boxed{M' = KM}$

where $K = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$ is "skew-symmetric" $\boxed{K^T = -K}$

Therefore,

$$\begin{aligned} Q' &= M' M^T + M (M')^T \\ &= K M M^T + M M^T K^T \\ &= KQ - QK \end{aligned}$$

Note that $Q(s) \equiv I$ is a solution to $(*)$.
By Fundamental uniqueness of ODEs, it must be the only solution. So $M M^T \equiv I$ $\forall s \in I$.
By continuity, $\det M \equiv 1$ $\forall s \in I$, so $M(s) \in SO(3)$ for all $s \in I$. This proves the claim.

Now, once we know

$$\{T(s), N(s), B(s)\} \text{ is a frame } \forall s \in I.$$

We can integrate $\alpha' = T$ to obtain. The rest of the details and the uniqueness part are left as an exercise.

$\square$

Remark on the condition $k > 0$

Otherwise, one may not be able to define the Frenet frame continuously. E.g. consider the trace of a curve

$$\{ y = x^3, \ z = 0 \} = \text{image}(\alpha)$$